# Changes in Students' Notation When Fractions Exceed One-Whole

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This study investigated the recordings of fraction notation, when the number of fractional parts exceeded the whole, made by 79 Year 4 students from two schools serving quite different communities. Approximately 30% of the Year 4 students changed the name of the denominator when the whole was exceeded by one unit-fraction; for example, the fraction name changed from quarters to fifths when creating 5 quarters. This practice of changing the name of the fractional part when the whole was exceeded was evident in both schools.

"The teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (Davis, Hunting, & Pearn, 1993, p. 63). Based on the performance of students on fractions questions in the National Assessment Program - Literacy and Numeracy (NAPLAN), the teaching of fractions remains a persistent problem in Australia. In the 2008 Year 7 non-calculator paper, over 70% of the cohort could not correctly interpret area models corresponding to three-quarters. Similarly, when asked to identify the remaining fraction of a potting mix, given it was $\frac{2}{3}$ soil and $\frac{1}{4}$ sand, over 157 000 Year 7 students, or 57% of the cohort, selected either $\frac{3}{7}$ or $\frac{4}{7}$ (Siek Toon, personal communication, 17 February 2010). That is, over half of all of the Year 7 students in Australia selected fraction answers based on adding numerators and denominators, a misinterpretation of the meaning of the fraction notation.

The unique nature of the fraction notation, representing both process and product, provides a substantial challenge to the teaching and learning of fractions (Steinle & Price, 2008; Yoshida & Kuriyama, 1995). The parts-of-a-whole explanation of "$\frac{2}{3}$" as "two out of three" that arises from working with partitioned models of fractions, becomes confusing when the whole is exceeded in needing to form "$\frac{4}{3}$." In a study of responses to fraction tasks involving 1676 students from over 90 classrooms across New South Wales, the introduction of questions involving the fraction notation increased the variety of incorrect interpretations of fractions (Gould, 2008). This reinforces the concern that the fraction notation may influence a number of fraction misconceptions (Pearn & Stephens, 2004).

Accounts of students transforming the identities of fractional parts when the whole is exceeded have come from teaching experiments involving iterating fraction parts in a computer environment (Tzur, 1999). When some students iterated a unit fraction beyond the whole, they changed the name of the fraction; for example from fifths to sixths. However, these teaching experiments were not focused on students' use of notation. Hence, there were two main foci for this study. First it sought to determine if renaming of fraction parts when the whole is exceeded occurs naturally outside of intensive teaching experiments. Second, it sought to investigate how students used fraction notation to record improper fractions.

To gain a better understanding of the role that fraction notation plays in the formation of the concept of a fraction as a quantitative measure, the accommodations Year 4 students make to their use of the fraction notation when fractions exceed one-whole were examined in two schools serving quite different communities.

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## Linking Notation to Part-Whole Models of Fractions

Common fractions are frequently introduced to students in Australia through contexts such as sharing food (Way & Bobis, 2011). In classrooms, shading partitions of shapes such as circles and squares often follows discussions of what constitutes "half an apple" or "a quarter of a sandwich." Shapes such as circles or squares are frequently used as models of the "whole" when describing the relationship between the parts and the whole. Within this paper I distinguish between the *interpretations* of fractions, and the *models* used to introduce fractions. Models are often used to represent mathematical ideas. Three common fraction models typical of school textbooks are the *linear* or *length model*, the *area model* and the *discrete* or *set model* (Watanabe, 2002). It is important not to confuse the various interpretations of fractions with the fraction models themselves, as the part-whole fraction interpretation, for example, can be applied to each of these models. The term *fraction model* is used in this paper to refer to something used in teaching to present the mathematical entity of a fraction. However, the way that a student chooses to interpret a given fraction model will vary.

When fraction notation is introduced in classrooms, it is used as a way of referencing a part-whole interpretation of a model (Boulet, 1998). That is, the notation used for fractions is intended to initially point to or index physical objects and, ultimately, the mathematical object of a single relational number. This dual use of the fraction notation, initially indexing physical objects as parts of a whole as well as a single number, passes without comment in classrooms. The ambiguity associated with the use of the fraction notation may in turn contribute to the fragile grasp many students have of the mathematically powerful notation, as a child's understanding of either fraction notation or part-whole relations does not ensure understanding of the other (Saxe, Taylor, McIntosh, & Gearhart, 2005).

Hiebert (1989) noted that written symbols function both as records of things already known and as tools for thinking. He argued that one of the factors contributing to students' poor performance on fractions was the tenuous connection between the form of fractions and a robust conceptual understanding of fractions. It is clear that the links students make between the meanings of the fraction symbols and understanding fractions as mathematical objects are often very weak (Charalambous & Pitta-Pantazi, 2007; Gould, 2005; Wong & Evans, 2008).

The parts-out-of-a-whole interpretation of the "$\frac{a}{b}$" notation may give meaning to "$\frac{2}{3}$" as "two parts out of three," but unless the notation is divorced from this way of referencing the context, "$\frac{4}{3}$" does not make sense.

## Different Ways of Interpreting Fraction Models

Using simple part-whole interpretations of fraction models as the primary means of defining fractions has clear limitations (Freudenthal, 1983) because a student may focus on numeric values associated with discrete parts, rather than the relationship between the parts and the whole. As Kieren (1988) explained:

> Because part-whole models of fractions conveniently help produce fractional language, the school mathematics fraction language of teacher and texts alike tend to orient a student to a static double count image and knowledge of fractions. The child, while being able to produce "correct" answers to questions, develops a mental model which is inappropriately inclusive (parts of a whole), rather than a powerful measure of inclusion (comparison to a unit) … (p. 177)

A focus on the parts that make up the whole can lead to an additive interpretation of the fraction notation, rather than a multiplicative comparison of the part to the whole. Many of the problems associated with learning fractions have been attributed to teaching efforts that have focussed almost exclusively on the part-whole interpretation of fractions (Streefland, 1991). Despite the limitations of part-whole interpretations of fraction models, many curricular offerings emphasise part-whole interpretations of fraction models almost exclusively in the primary years (Middleton, Toluk, deSilva, & Mitchell, 2001).

### *Fraction Schemes and Operating with Units*

In recent years there has been a focus on describing schemes that support children's development of fraction-based reasoning (Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Tzur, 2000). The various schemes have been proposed as models of students' thinking. Central to describing these schemes is the way that students operate with units and coordinate units in giving meaning to fractional quantities (Hackenberg, 2007; Watanabe, 1995). The schemes used to characterise students' thinking include the *simultaneous partitioning scheme*, the *part-whole scheme*, the *equi-partitioning scheme*, the *partitive fractional scheme*, the *reversible partitive fractional* scheme and the *iterative fractional scheme* (Norton, 2008). The iterative fraction scheme includes improper fractions. Tzur (2000) describes the iterative fraction scheme as requiring the transformation of the iterable part

> "…into an *invariant, multiplicative relation* between the size of an iterable unit fraction and the whole. Here, a fraction word ("one-sixth") and a fraction numeral ("1/6") symbolize for the child the size of a unit fraction that maintains a particular (1-to-6) relation to the whole regardless of how this unit was produced (e.g., divided a whole into six parts) or which operations the child performed on it (e.g., iterated 1/6 seven times or added 5/6 + 4/6). Put differently, the child does not lose sight of the relative size of the unit fraction while using her or his number knowledge to operate on it. (p. 143)

In a teaching experiment with two fourth-graders designed to explain children's conceptions of fractions on the basis of iteration of units, Tzur (1999) noted that an unusual thing occurred when the students attempted to produce an improper fraction; that is, a fraction greater than one, via iteration of a unit fraction. They correctly regarded the non-unit fraction produced by iterating, say, $\frac{1}{5}$ four times as $\frac{4}{5}$, but when they iterated the same $\frac{1}{5}$ six times, they thought of the result as $\frac{6}{6}$ and each part was then regarded as $\frac{1}{6}$. The students had been engaging with these iteration tasks in a computer microworld. When the process of unit iteration produced a set larger than the reference whole, it was as if it were regarded as a different set.

Describing what happens when students attempt to produce improper fractions through unit iteration was addressed by Hackenberg (2007) in a year-long teaching experiment involving four sixth-grade students. One of the tasks analysed involved asking two of the students to draw seven-fifths of a candy bar, given the drawing of the rectangle on their paper represented one candy bar. Although both girls correctly created seven-fifths of the rectangle with one of the girls stating that she knew that seven-fifths was one and two-fifths, when asked about the size of the pieces in the bars they had drawn, the girls maintained that the pieces were sevenths. Clearly something unexpected was occurring when a fraction part was iterated beyond the whole.

## Method

The preceding descriptions of students transforming the identities of fractional parts when the whole is exceeded have come from teaching experiments involving iterating fraction parts in a computer environment. This study was designed to determine if this transformation of the identities of fractional parts could be detected in students' fraction notation when students had not engaged with unit iteration in computer microworlds.

### *Participants and Tasks*

Year 4 classes in two primary schools were selected to complete two tasks. The two schools served markedly different communities. The Index of Community Socio-Educational Advantage (ICSEA) for one school was more than 1 standard deviation below the ICSEA mean while the other school was 1.75 standard deviations above the mean. These measures of socio-educational advantage are sufficiently far apart (over 2 standard deviations) as to reflect real differences between the school communities. Two classes provided 48 responses from the low ICSEA school and one complete class and the Year 4 component of a composite class provided 31 responses from the school with the high ICSEA measure.

In each task the students were presented with a drawn rectangle and told that the drawing represented a piece of chocolate. They were then asked to draw a piece of chocolate that was five-quarters in the first, and four-thirds in the second, the size of the representation of the piece of chocolate. The fractions in the questions were written in words and the classroom teachers could read the questions to their students. Students work with models of fractions with denominators 2, 4, and 8 before the end of Year 2 but are not formally introduced to thirds until Year 5.

## Results and Discussion

The size of the units the students formed was recorded, along with the notation used to record the fraction units. Using only the word "quarter" to describe the size of the fraction pieces was considered a separate category of answer (Table 1) because this term was used in the question. The idea that five-quarters can become five-fifths was not restricted to one school, with approximately 30% of the Year 4 students at each school recording five-quarters as fifths.

Table 1
*Year 4 students' fraction labels when drawing five-quarters*

| Notation | High ICSEA | Low ICSEA | Total |
|---|---|---|---|
| Named fraction as $\frac{1}{4}$ | 12 | 14 | 26 |
| Named fraction as $\frac{1}{5}$ | 11 | 14 | 25 |
| Named fraction "quarters" | 2 | 11 | 13 |
| Omitted fraction name | 3 | 7 | 10 |
| Other | 3 | 2 | 5 |
| Total | 31 | 48 | 79 |

"Other" responses included labelling the fraction parts with counting numbers (two students, one from each school), inverting the fraction notation scheme and answering "$\frac{5}{1}$" (one student), answering "1.5 cm" which corresponded to one-quarter of the length of the original fraction bar (one student), and writing "$\frac{3}{4}$" (one student).

When asked to draw a piece of chocolate five-quarters the size of the one presented, almost the same number of Year 4 students used the fraction notation for fifths as used the notation for quarters.

Naming the five quarters as fifths does not appear to be influenced by the difference between the ICSEA measures for these two schools. Using a chi-square test to compare results from the high ICSEA school with the low ICSEA school, a 0.05 level of significance and one degree of freedom, any hypothesized difference between the two schools failed to produce a statistically significant value, $\chi^2$ (1, $N$ = 79) = 0.35, p = 0.56.

## *Transformed fraction identities and measurement*

The renaming of the fraction parts (from $\frac{1}{4}$ to $\frac{1}{5}$) did not appear to be influenced by the measurement skills of the students. In Figure 1 the answer is apparently formed without attention to the length of the original unit, as the result is approximately half the length of the initial drawing of the chocolate.

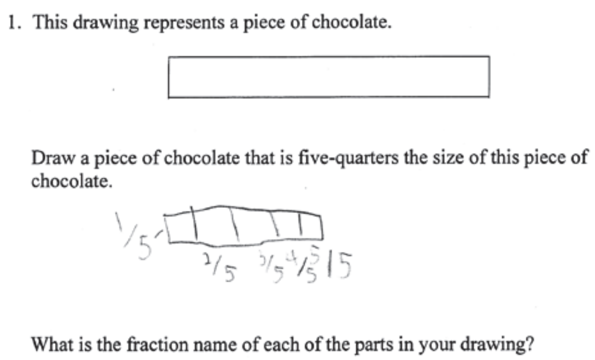


*Figure 1.* Each quarter is recorded as one-fifth.

Even when the diagram was carefully drawn to the correct size (Figure 2), the nominated value of the fractional parts could still be transformed.

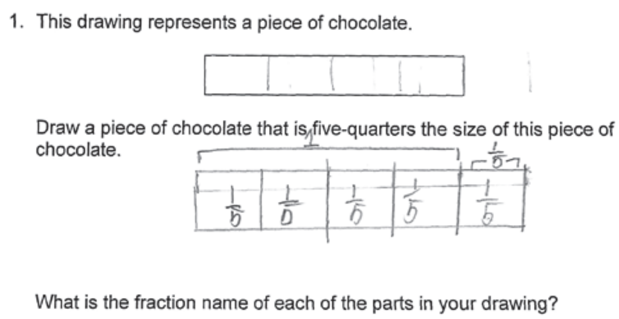


*Figure 2.* Each quarter of the original rectangle recreated and labelled as fifths.

The initial rectangle was partitioned into four quarters. The size of the quarters was effectively replicated in producing the five-quarters requested, but with the names of the fractional parts changing to fifths. That is, the size relationships of the fractional parts appear to be maintained but the notational tags have changed. What started as quarters has been given a new name when the quantity exceeded one whole-unit (with four one-fifths clearly forming the whole).

### *Creating a new "whole"*

Sometimes a *whole* corresponding to five-fifths appeared to be created (Figure 3). Here, increasing the size by a quarter resulted in a diagram larger than the initial representation of the chocolate, accompanied by notation showing a new accumulation of parts (in a similar way to the response in Figure 1).

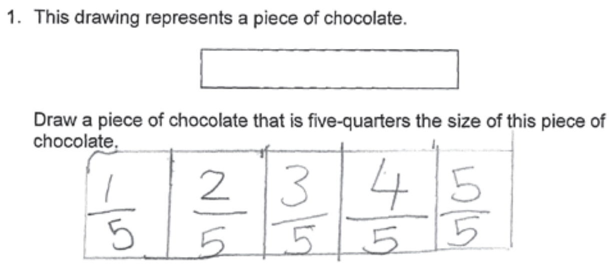


*Figure 3*. Five quarters recorded as fifths.

Comparing Figure 1 with Figure 3, the length of the representation does not appear to influence the change in notation. The attribute of length may not be essential to a student's use of an apparently linear model to represent the whole.

Kieran's depiction of the notational change when the whole is exceeded (Figure 4) was unique.

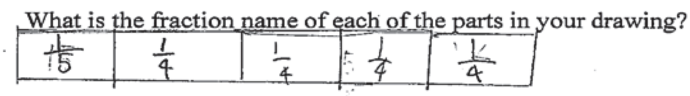


*Figure 4.* A change in the fraction notation beyond one-whole.

For Kieran, four units of $\frac{1}{4}$ still appeared to represent the whole, but the fifth-quarter was recorded as $\frac{1}{5}$. This response captured what might be a transition in Kieran's thinking. That is, the notation associated with the iteration of the $\frac{1}{4}$-unit suggests that four-quarters still represent the whole but the fifth quarter is now one of five equal parts and so earns the notation of $\frac{1}{5}$.

The transformation that takes place when the number of fractional units exceeds the whole can also create a new whole composed of five-fifths (Figure 5), as well as an additional fractional part retaining the notational label $\frac{1}{5}$.

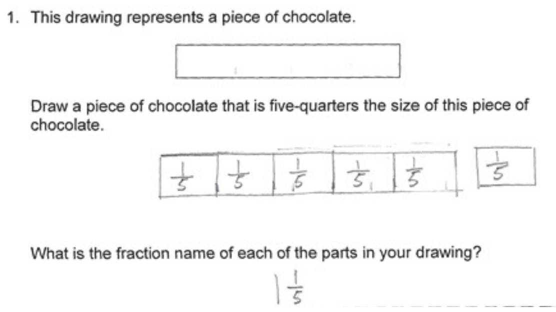


*Figure 5.* Five-fifths forming the whole and the additional part described as one-fifth.

The second task also saw a substantial proportion of the responses recording a transformation of the fraction name when the whole was exceeded (Table 2).

Table 2
*Year 4 students' fraction labels when drawing four-thirds*

| Notation | High ICSEA | Low ICSEA | Total |
|---|---|---|---|
| Named fraction as $\frac{1}{3}$ | 13 | 18 | 31 |
| Named fraction as $\frac{1}{4}$ | 11 | 11 | 22 |
| Omitted fraction name | 6 | 10 | 16 |
| Other | 1 | 6 | 7 |
| Named fraction "thirds" | 0 | 3 | 3 |
| Total | 31 | 48 | 79 |

Approximately 27% of the Year 4 students used notation indicating a change in the name of the fraction parts to quarters when they were asked to represent four-thirds. The use of the written form of the answer "thirds" was less common than the use of the written form in the first task. Using Fischer's exact test to compare the use of notation indicating transformed fraction names from thirds to quarters between the two schools (high & low ICSEA) failed to produce a statistically significant value (two-tailed $p = 0.30$).

If Kieran's notation (Figure 4) was in transition in Question 1, it showed further development in Question 2 (Figure 6).

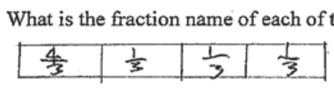


*Figure 6.* Kieran's evolving fraction notation.

This can be interpreted as an accumulation of thirds from right to left, with the final entry showing four-thirds. That is, the three units on the right are recorded as thirds and form a whole, but the fourth unit shows a fraction notation that could attempt to record the fourth third. How the whole is treated in students' recordings and notation is a good indication of their struggle to address the sophisticated use of units at different levels required when a unit fraction is iterated beyond the whole (Hackenberg, 2007).

A student's representation of fractions reflects the coordination of knowledge of notational conventions with particular kinds of unit-based relations. However, symbolised fraction words and notations are not identical with understanding of coordinating units at different levels. The dual use of the fraction notation, initially indexing physical objects as parts of a whole as well as a single number, appears to have created an unexpected (and largely undetected) transformed use of the notation when the whole is exceeded. Clearly, the effectiveness of the models used to teach fractions and the ways students interpret them needs further research.

## Practical Implications for Teaching

The *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority, 2012) sets the expectation that students should deal with improper fractions involving halves, quarters, and thirds in Year 4. Yet many Year 4 students struggle with maintaining the link between the fraction name as captured in their use of notation, and the whole when the whole is exceeded. The fraction appears to change "families."

### *Changing the way fraction notation is introduced*

Teaching experiences need to provide opportunities for students to come to know fractions as more than simply "counting." However, the textbooks students and teachers engage with often start by introducing fraction symbols, and then presenting a meaning for the symbol, as in Figure 7.

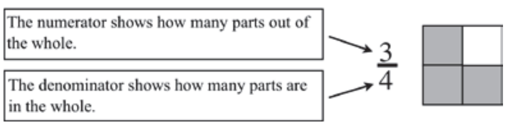


*Figure 7*. A textbook definition of fraction symbols.

The practice of introducing fraction notation as corresponding to two simple counts or "x parts *out of* y equal parts" has many obvious limitations. Fractions depict multiplicative relationships not additive counts. Rather than emphasising the countable features of regional models, teaching benefits from beginning by building meaning. That is, teaching fractions should start with problems involving sharing, slicing, and distributing rather than fraction symbols. When the unit names associated with the multiplicative relationship between a partitioning and the whole have been established, recording schemes can be gradually introduced.

Streefland (1991, p. 51) used problems of sharing different quantities of pizza on tables with different numbers of customers to introduce progressively simplified seating diagrams. He used this context to co-construct a system of recording as a transition towards the fraction notation (Figure 8).

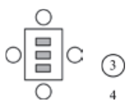


*Figure 8.* A seating arrangement with a "table symbol."

As well as questions involving sharing, say, 3 lamington fingers among 4 people, equivalence can be introduced with questions such as: "Do you get more, less, or the same if 6 lamington fingers on a table are shared equally among 8 people?" The process of division, as the inverse of multiplication, is essential in creating multiplicatively related fraction quantities. Improper fractions involving quarters can develop from arrangements such as a table with 5 lamington fingers that are shared equally among 4 people. Drawing what each person would receive is a helpful way of linking the measurement sense of fraction units with the "table symbol."

Dealing with fractions greater than one can also emphasise the multiplicative relationship between the part and the whole. Mathematics ultimately requires abstract representation, but young children understand such representation more readily if it is derived from meaningful experience than if it results from learning definitions and rules.

### *Introducing the constant whole*

If students are to develop an understanding of fractions as mathematical objects, the idea that fractions reference a constant whole needs to be developed. This idea is typically absent from current curriculum documents, or, at best, is only implied. Students need opportunities to recognise that partitioned fractions are always dependent upon the whole of which they are part. It is essential to establish the idea of the *equal whole* (or universal "one") as part of the concept image of students before any meaning can be given to operations on fractions. The symbolism $\frac{1}{2}+\frac{1}{4}$ has no meaning without reference to a universal equal-whole.

In the early years, students can be asked to share fairly two similar but unequal lengths of liquorice between two people. That is, they are able to use two different sized units. In later years, reconstructing the whole from a non-unit part can make the whole explicit. For example, students could be provided with a tower of six connected blocks, and asked to indicate how high the full tower would be if what they had was three-quarters of the whole tower.

### *Linking the fraction notation to division contexts*

Rather than starting with the symbols, fractions should be introduced through equal sharing contexts that use countable continuous quantities that can be cut or divided. These problem contexts yield results equivalent to improper fractions as well as proper fractions. However, the real challenge remains in linking the meaning of the fraction notation to the problem contexts. A notation that is closer to representing the results of an indicated division, such as Streefland's table symbol, may be more helpful than a notation defined as "x parts out of y equal parts." If we want students to engage with improper fractions, they need opportunities that move beyond associating the name of the fraction family, the denominator, only with a number of parts.

## References

Australian Curriculum Assessment and Reporting Authority. (2012). *The Australian Curriculum: Mathematics*. Retrieved 8 February 2012, from http://www.australiancurriculum.edu.au/Download

Boulet, G. (1998). Didactical implications of children's difficulties in learning the fraction concept. *Focus on Learning Problems in Mathematics, 21*(3), 48-66.

Charalambous, C., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fraction. *Educational Studies in Mathematics, 64*, 293-316.

Davis, G., Hunting, R. P., & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior, 12*, 63-76.

Freudenthal, H. (1983). *A didactical phenomenology of mathematics*. Boston: Kluwer.

Gould, P. (2005). Year 6 students' methods of comparing the size of fractions. In P. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce, & A. Roche (Eds.), *Building connections: Theory, research and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia), (pp. 393-400). RMIT Melbourne: MERGA.

Gould, P. (2008). *Children's* quantitative *sense of fractions.* Unpublished PhD Thesis, Macquarie University, Sydney.

Hackenberg, A. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior, 26*(1), 27-47.

Hackenberg, A., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior, 28*, 1-18.

Hiebert, J. (1989). The struggle to link written symbols with understandings: An update. *Arithmetic Teacher, 36*(7), 38-44.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert, & M. J. Behr (Eds.), *Number concepts and operations in the middle grades* (Vol. 2), pp. 162-181). Reston, VA: National Council of Teachers of Mathematics.

Middleton, J. A., Toluk, Z., deSilva, T., & Mitchell, W. (2001). The emergence of quotient understandings in a fifth-grade classroom: A classroom teaching experiment. In R. S. Speiser, & C. Maher (Eds.), *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 263-271). Snowbird, Utah: ERIC.

Norton, A. (2008). Josh's operational conjectures: Abductions of a splitting operation and the construction of new fractional schemes. *Journal for Research in Mathematics Education, 39*(4), 401-430.

Pearn, C., & Stephens, M. (2004). Why do you have to probe to discover what Year 8 students really think about fractions? In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics education for the third millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia), (pp. 430-437). Sydney: MERGA.

Saxe, G. B., Taylor, E. V ., McIntosh, C., & Gearhart, M. (2005). Representing fractions with standard notation: A developmental analysis. *Journal for Research in Mathematics Education, 36*(2), 137-157.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Steinle, V., & Price, B. (2008). What does three-quarters look like? Students' representations of three-quarters. In M. G oos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia), (pp. 483–489). Brisbane: MERGA.

Streefland, L. (1991). *Fractions in realistic mathematics education. A paradigm of developmental research.* Dordrecht: Kluwer Academic.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Tzur, R. (2000). An integrated research on children's construction of meaningful, symbolic, partitioning-related conceptions, and the teacher's role in fostering that learning. *Journal of Mathematical Behavior, 18*(2), 123-147.

Watanabe, T. (1995). Coordination of units and understanding of simple fractions: Case studies. *Mathematics Education Research Journal, 7*(2), 160-175.

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*, 457-463.

Way, J., & Bobis, J. (Eds.). (2011). *Fractions: Teaching for understanding*. Adelaide: The Australian Association of Mathematics Teachers.

Wong, M., & Evans, D. (2008). Fractions as a measure. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia), (pp. 597-603). Brisbane: MERGA.

Yoshida, H., & Kuriyama, K. (1995). Linking meaning of symbols of fractions to problem situations. *Japanese Psychological Research, 37*, 229-239.